Int. J. Heat Mass Transfer. Vol. 16, pp. 1037–1043. Pergamon Press 1973. Printed in Great Britain

INWARD SPHERICAL SOLIDIFICATION—SOLUTION BY THE METHOD OF STRAINED COORDINATES

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(*Received* 21 *July* 1972)

Abstract—The method of strained coordinates is applied to obtain a perturbation solution for spherical solidification of a saturated liquid. The solution is uniformly valid and can be applied as the freezing front approaches the center where the regular-perturbation solution is found to diverge. The wall temperature is assumed constant; however, the technique should also be applicable to other types of boundary conditions. The properties of the solidified material are assumed to be constant. A non-linear transformation is applied to the sequence of partial sums in the perturbation solution to increase its range of applicability. The solutions obtained are compared with numerical results.

NOMENCLATURE

- c, specific heat of the solidified material;
- k, thermal conductivity of the solidified material;
- L, latent heat of fusion;
- R, radial position in the solidified material;
- R_f, radial position of the freezing front;
- R_0, radial position of the fixed (spherical) boundary;
- T, temperature distribution in the solidified material;
- T_f, freezing temperature;
- T_0, temperature at the fixed (spherical) boundary;
- t, time;
- α, thermal diffusivity of the solidified material, $k/(\rho c)$;
- ρ, density of the solidified material.

Dimensionless quantities

- r, normalized position in the solidified material, R/R_0;
- r_f, normalized freezing-front position, R_f/R_0;
- u, normalized temperature distribution in the solidified material, $(T - T_0)/(T_f - T_0)$;
- u_i, coefficient of ε^i in the power-series expansion of u;
- ε, perturbation physical parameter, $c(T_f - T_0)/L$;
- σ_i, coefficient of ε^i in the power-series expansion of r;
- τ, normalized time, $k(T_f - T_0)t/(\rho L R_0^2)$;
- ϕ, Ψ, independent variables introduced in equations (6).

INTRODUCTION

IN A PREVIOUS paper [1], a regular parameter-perturbation technique was introduced for outward and partial inward spherical solidification. The boundary conditions were the same as those presently considered in this paper. The solution was found to diverge for inward solidification as the freezing front approached the center of the sphere. The series solution was then modified to insure convergence; however,

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the temperature distributions so obtained had a point of singularity at the center and hence were inaccurate in its vicinity. Numerical solutions of the same problem were obtained by Tao [2] who, for inward solidification, presented the results in graphical form. These results were then tabulated in [3].

The objective of this paper is to introduce a parameter-perturbation technique which yields a solution uniformly valid for spherical solidification. Although the method of strained coordinates is not recommended [4] for parabolic differential equations, it will be used here and shown to yield good solutions. The non-linear transformation of Shanks [5] will be applied to the sequence of partial sums in the perturbation solution to considerably increase its range of applicability.

The solutions obtained will apply for outward as well as inward spherical solidification. However, the regular-perturbation solution of [1] is simpler and more straightforward, therefore, it should be used in the case of outward solidification. The freezing temperature and all other properties will be assumed constant. Further, the liquid will be assumed to be at the freezing temperature. The wall temperature will be assumed to be constant. For melting problems, the solutions presented will apply if the melt is assumed to remain stationary. The perturbation technique introduced in this paper should also be applicable to other types of boundary conditions and in the cylindrical geometry.

ANALYSIS

Consider the one-dimensional spherical configuration shown in Fig. 1. The heat flow within the frozen spherical shell is governed by the transient heat conduction equation. For spherical symmetry and with constant properties, this has the form

$$\frac{\partial T}{\partial t} = \frac{\alpha}{R}\frac{\partial^2(RT)}{\partial R^2} \tag{1}$$

The constant temperatures at the fixed boundary,

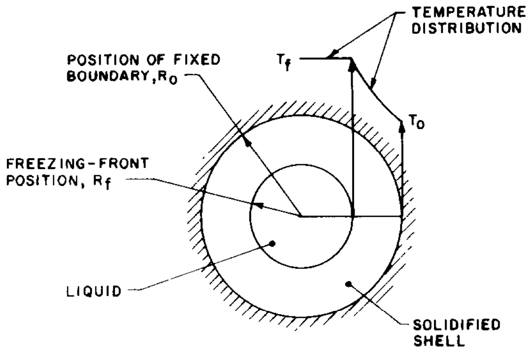

FIG. 1. One-dimensional spherical geometry shown for the case of inward solidification.

T_0, and at the freezing front, T_f, yield the boundary conditions:

$$T(t, R = R_0) = T_0, \qquad T(t, R = R_f) = T_f. \tag{2}$$

An energy balance at the interface, with the temperature of the liquid assumed constant at the freezing temperature, yields

$$\frac{\mathrm{d}R_f}{\mathrm{d}t} = \frac{k}{\rho L}\left.\frac{\partial T}{\partial R}\right|_{R=R_f}. \tag{3}$$

Define the dimensionless parameter, ε, temperature, u, radial position, r, freezing-front position, r_f, and time, τ, as

$$\varepsilon = c\frac{T_f - T_0}{L}, \qquad u = \frac{T - T_0}{T_f - T_0}, \qquad r = \frac{R}{R_0}$$

$$r_f = \frac{R_f}{R_0}, \qquad \tau = k\frac{T_f - T_0}{\rho L R_0^2}t. \tag{4}$$

The physical parameter, ε, is a (qualitative) measure of the sensible heat in the solidified material relative to the latent heat of fusion. Combine equations (1)–(4) and change variables $(t, r) \to (r_f, r)$ to obtain the normalized form of the boundary-value problem:

$$\varepsilon\frac{\partial u}{\partial r_f}\left.\frac{\partial u}{\partial r}\right|_{r=r_f} = \frac{1}{r}\frac{\partial^2(ru)}{\partial r^2}$$

$$u(r_f, r = 1) = 0, \qquad u(r_f, r = r_f) = 1 \tag{5}$$

$$\frac{\mathrm{d}r_f}{\mathrm{d}\tau} = \left.\frac{\partial u}{\partial r}\right|_{r=r_f}.$$

The regular-perturbation solution to the system of equations (5) was obtained in [1] and is presented in the Appendix. This solution is clearly divergent (for any non-zero value of ε) as the freezing front approaches the center of the sphere.

Equations (5) will now be solved by straining the coordinates r_f and r. Define two new independent variables:

$$\phi = \phi(r_f, r), \qquad \Psi = \phi(r_s, r_f) \tag{6}$$

and expand the variables r and r_f as

$$\begin{aligned} r &= \phi + \varepsilon\sigma_1(\Psi, \phi) + \varepsilon^2\sigma_2(\Psi, \phi) + \dots \\ r_f &= \Psi + \varepsilon\sigma_1(\Psi, \Psi) + \varepsilon^2\sigma_2(\Psi, \Psi) + \dots \end{aligned} \tag{7}$$

where the functions $\sigma_1(\phi, \Psi)$ have been introduced and will be determined in the solution. The normalized temperature distribution taken as a function of the variables ϕ, Ψ and of the parameter ε can be expanded as

$$u(\Psi, \phi; \varepsilon) = u_0(\Psi, \phi) + \varepsilon u_1(\Psi, \phi) + \varepsilon^2 u_2(\Psi, \phi) + \dots \tag{8}$$

The following conditions are established in order to satisfy the boundary condition at the fixed wall and the initial freezing-front position:

$$\lim_{\phi \to 1} \sigma_i(\Psi, \phi) = \lim_{\Psi \to 1} \sigma_i(\Psi, \Psi) = 0. \tag{9}$$

Change variables $(r_f, r) \to (\Psi, \phi)$ in equations (5) and expand in terms of ε according to equations (7) and (8), to obtain for the zeroth-order boundary-value problem:

$$\frac{\partial^2(\phi u_0)}{\partial \phi^2} = 0$$
$$u_0(\Psi, \phi = 1) = 0, \qquad u_0(\Psi, \phi = \Psi) = 1 \tag{10}$$

which can be solved to obtain

$$u_0 = \frac{(1/\phi) - 1}{(1/\Psi) - 1}. \tag{11}$$

The first-order boundary-value problem can then be written as

$$\frac{\partial^2}{\partial \phi^2}\left(\phi u_1 + \frac{\Psi}{1-\Psi}\frac{\sigma_1}{\phi}\right) = -\frac{1-\phi}{\Psi(1-\Psi)^3}$$
$$u_1(\Psi, \phi = 1) = u_1(\Psi, \phi = \Psi) = 0. \tag{12}$$

It is desired, if possible, to choose the function σ_1 such that the first-order term in the temperature distribution, u_1, is identically equal to zero. The simplest form of σ_1 to accomplish this is

$$\sigma_1 = -\frac{\phi(1-\phi)^3}{6\Psi^2(1-\Psi)^2}. \tag{13}$$

The second-order boundary-value problem then becomes

$$\frac{\partial^2}{\partial \phi^2}\left(\phi u_2 + \frac{\Psi}{1-\Psi}\frac{\sigma_2}{\phi}\right) = \frac{2}{3}\frac{1-\phi}{\Psi^2(1-\Psi)^3} + \frac{(4 + 4\Psi - 15\phi)(1-\phi)^3}{6\Psi^3(1-\Psi)^5} \tag{14}$$
$$u_2(\Psi, \phi = 1) = u_2(\Psi, \phi = \Psi) = 0.$$

The simplest function σ_2 to make u_2 identically zero is

$$\sigma_2 = \left[\frac{1}{9} - \frac{1 - 4\Psi + 10\phi}{120}\frac{(1-\phi)^2}{\Psi(1-\Psi)^2}\right] \times \frac{\phi(1-\phi)^3}{\Psi^3(1-\Psi)^2}. \tag{15}$$

The second equation in (7) can now be combined with equations (13) and (15); yielding for the normalized freezing-front position:

$$r_f = \Psi - \varepsilon\frac{1-\Psi}{6\Psi} + \varepsilon^2\frac{(22\Psi - 3)(1-\Psi)}{360\Psi^3}. \tag{16}$$

Express the energy balance at the freezing front in terms of ϕ and Ψ to obtain in expanded form:

$$\frac{d\tau}{d\Psi} = \frac{1}{\left.\dfrac{\partial u_0}{\partial \phi}\right|_{\phi=\Psi}}\left\{1 + \varepsilon\left[\frac{d\sigma_1(\Psi, \Psi)}{d\Psi} + \left.\frac{\partial \sigma_1}{\partial \phi}\right|_{\phi=\Psi}\right]\right.$$

$$+\varepsilon^2\left[\frac{d\sigma_2(\Psi,\Psi)}{d\Psi}+\frac{\partial\sigma_2}{\partial\phi}\bigg|_{\phi=\Psi}+\frac{d\sigma_1(\Psi,\Psi)}{d\Psi}\frac{\partial\sigma_1}{\partial\phi}\bigg|_{\phi=\Psi}\right]+\dots\Bigg\} \quad (17)$$

which can be shown to simplify to

$$\tau=\frac{3(1-\Psi)^2-2(1-\Psi)^3}{6}+\varepsilon\frac{(1-\Psi)^2}{3}-\varepsilon^2\frac{(1-\Psi)^2}{180\Psi^2}. \quad (18)$$

The solutions presented above are implicit relations for the temperature distribution, u, and the freezing time, τ, in terms of the variables r and r_f. The freezing-time solution including two terms can easily be expressed explicitly in terms of r_f if

$$\Psi=\frac{6r_f-\varepsilon+\sqrt{[(6r_f-\varepsilon)^2+24\varepsilon]}}{12} \quad (19)$$

is substituted into the first two terms in equation (18).

The first three terms of the perturbation solution have been presented above. It should be possible to calculate more terms in the solution; however, this procedure becomes algebraically lengthy. The terms so far calculated can yield, if properly used, a large amount of information. In order to accomplish this, the non-linear transformations of Shanks [5] can be used. The transformations become most useful when the perturbation series is slowly convergent or divergent. Since only the first three terms have been calculated in the above solution, it is possible to apply but the simplest of these transformations. For the normalized freezing-front position in equations (7), the result is

$$r_f^*=\frac{\Psi\sigma_1(\Psi,\Psi)-\varepsilon[\Psi\sigma_2(\Psi,\Psi)-\sigma_1^2(\Psi,\Psi)]}{\sigma_1(\Psi,\Psi)-\varepsilon\sigma_2(\Psi,\Psi)} \quad (20)$$

which can be simplified to

$$r_f^*=\Psi\frac{60\Psi^2+\varepsilon(32\Psi-13)}{60\Psi^2+\varepsilon(22\Psi-3)}. \quad (21)$$

Application of the same transformation to the series for the freezing time, equation (18), yields

$$\tau^*=\frac{[60\Psi^2(1+2\Psi)+\varepsilon(1+2\Psi+120\Psi^2)](1-\Psi)^2}{6(60\Psi^2+\varepsilon)}. \quad (22)$$

The implicit relation between r_f^* and τ^* is given by the above two equations. Application of the same transformation to the normalized position in equation (7) gives

$$r^*=\phi\frac{\Psi+\frac{2}{3}\varepsilon\left[1-\frac{(1-\phi)^2(13-12\Psi+20\phi)}{40\Psi(1-\Psi)^2}\right]}{\Psi+\frac{2}{3}\varepsilon\left[1-\frac{3}{40}\frac{(1-4\Psi+10\phi)(1-\phi)^2}{\Psi(1-\Psi)^2}\right]} \quad (23)$$

Combining equations (11), (21) and (23), one obtains an implicit solution for the normalized temperature distribution in terms of r_f^* and r^*. Equation (21) can easily be inverted at the instant of freezing to the center, yielding

$$\Psi(r_f^*=0)=\frac{\sqrt{(64\varepsilon^2+195\varepsilon)}-8\varepsilon}{30}.$$

RESULTS AND DISCUSSION

Results from the above solutions have been presented graphically in Figs. 2–5. The normalized temperature distribution, equation (11), at the instant of freezing to the center is compared in Fig. 2 with the numerical results of Tao [3]. Values of 0·1 and 0·5 were assigned to the perturbation parameter, ε. The difference, for $\varepsilon = 0{\cdot}1$, between the perturbation and numerical solutions, we believe results from errors introduced in the numerical integration of the boundary-value problem as the freezing front approaches the center.

The freezing times including two and three terms in the series solution, equation (18), are shown in Fig. 3 for values of ε of 0·1, 0·5 and 1. Once again, there exists some disagreement for $\varepsilon = 0{\cdot}1$ between the perturbation and numerical solutions. The perturbation solution for this small value of ε should be accurate. For $\varepsilon = 1$ there is a large difference from the first to the

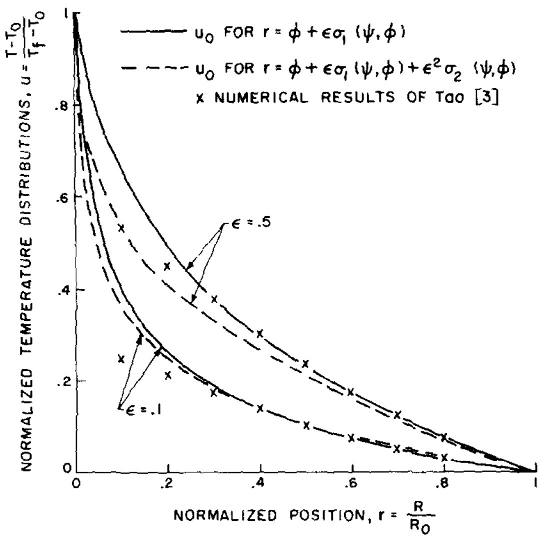

FIG. 2. Normalized temperature distributions at the instant of freezing to the centre ($r_f = 0$) and numerical results of Tao [3].

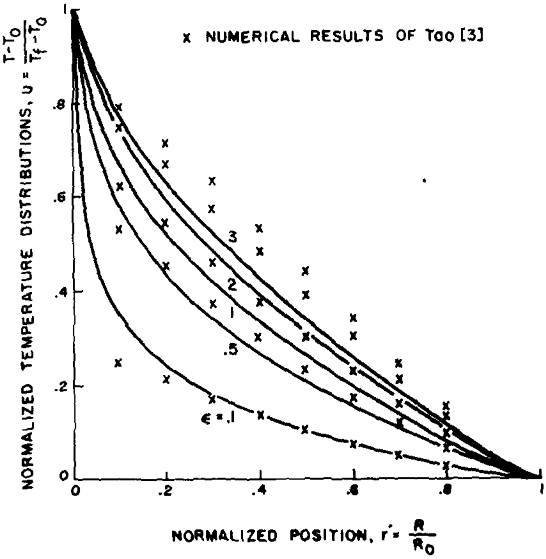

FIG. 4. Normalized temperature distributions after non-linear transformation of Shanks [5] at the instant of freezing to the center ($r_f^* = 0$) and numerical results of Tao [3].

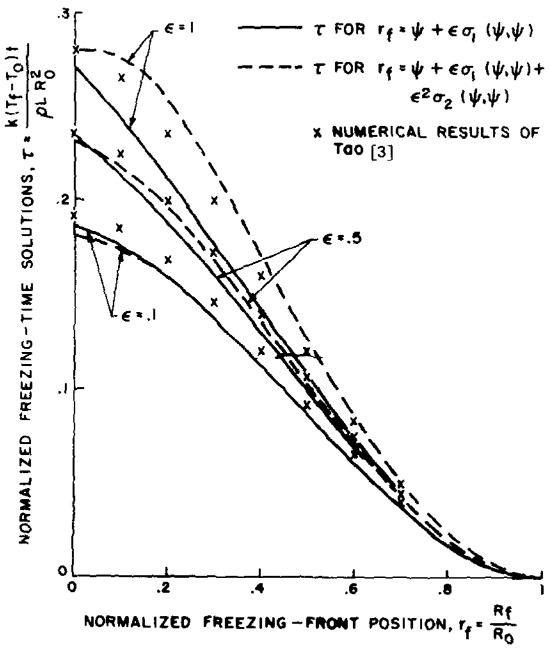

FIG. 3. Normalized freezing-time solutions for inward solidification and numerical results of Tao [3].

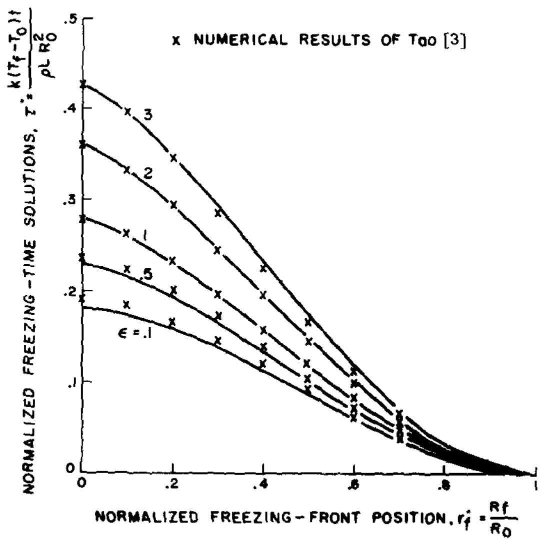

FIG. 5. Normalized freezing-time solutions after non-linear transformation of Shanks [5] for inward solidification and numerical results of Tao [3].

second-order perturbation solution; hence, the series will be either slowly convergent or divergent. It is likely that the latter condition exists, since the slope of the second-order solution becomes positive for freezing-front positions near the center.

Figures 4 and 5 show the non-linear transformed solutions for various values of ε up to $\varepsilon = 3$. The difference between the perturbation solution for the temperature distribution and [3] becomes large for $\varepsilon = 2$ and 3. It might seem logical to conclude that the non-linear transformed perturbation solution is in error for these moderate values of ε. However, the non-linear transformed freezing-time solution is in good agreement (see Fig. 5), even for $\varepsilon = 3$, with [3]. Note that the difference between our perturbation solution and the results of Tao [3], for the temperature distribution for $r_f = 0$, exists even for $\varepsilon = 0{\cdot}1$; in which case the perturbation solution seems to be rapidly convergent. Therefore, the possibility remains that large errors are introduced in the numerical integration of [3] as the freezing front approaches the center, resulting at this instant in erroneous temperature distributions. Since close to the center freezing rates are relatively high, this error should have a small effect on the freezing-time solutions.

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APPENDIX

Assume a solution to equations (5) of the form:

$$u(r_f, r; \varepsilon) = u_0(r_f, r) + \varepsilon u_1(r_f, r) + \varepsilon^2 u_2(r_f, r) + \ldots \qquad (A.1)$$

Substitution and equating coefficients of equal powers of ε yields for the temperature distribution:

$$u_0 = \frac{1 - (1/r)}{1 - (1/r_f)}$$

$$\frac{u}{u_0} = 1 + \frac{1}{6}\left[1 - \left(\frac{r}{r_f}\right)^2 u_0^2\right]\frac{\varepsilon}{r_f^2} - \left\{\frac{1}{36}\left[1 - \left(\frac{r}{r_f}\right)^2 u_0^2\right] + \frac{4r_f - 1}{120}\left[1 - \left(\frac{r}{r_f}\right)^4 u_0^4\right]\right\}\left(\frac{\varepsilon}{r_f^2}\right)^2 \qquad (A.2)$$

and for the freezing-time solution:

$$\tau = \frac{3(r_f - 1)^2 + 2(r_f - 1)^3}{6} + \frac{(r_f - 1)^2}{6}\varepsilon - \frac{1}{45} \times \frac{(r_f - 1)^2}{r_f}\varepsilon^2. \qquad (A.3)$$

SOLIDIFICATION SPHERIQUE VERS L'INTERIEUR—SOLUTION PAR LA METHODE DES COORDONNEES DEFORMEES.

Résumé—On applique la méthode des coordonnées déformées pour obtenir une solution de perturbation pour la solidification sphérique d'un liquide saturé. La solution est uniformément valide et peut être appliquée quand le front de solidification approche le centre où la solution de perturbation régulière diverge. La température est supposée constante à la paroi; cependant, la technique devrait aussi être applicable à d'autres types de conditions aux limites. Les propriétés du matériau solidifié sont supposées constantes. On a appliqué une transformation non linéaire à la suite de sommes partielles dans la solution de perturbation pour augmenter son domaine d'applicabilité. Les solutions obtenues sont comparées aux résultats numériques.

KUGELFÖRMIGE VERFESTIGUNG, LÖSUNG MIT DER METHODE TRANSFORMIERTER KOORDINATEN

Zusammenfassung—Die Methode der transformierten Koodinaten wurde angewandt, um eine Störungslösung für kugelförmige Verfestigung einer gesättigten Flüssigkeit zu erhalten. Die Lösung ist überall gültig und kann auch benutzt werden, wenn sich die Gefrierfront dem Mittelpunkt nähert, wo die reguläre Störungslösung divergiert.

Die Wandtemperatur ist konstant vorausgesetzt, jedoch könnte die Methode auch für andere Kanalbedingungen anwendbar sein.

Konstante Stoffeigenschaften des verfestigten Materials werden vorausgesetzt. Eine nichtlineare Transformation wurde auf die Folge von Teilsummen in der Strömungslösung angewandt, um die Gültigkeit zu erweitern. Die so erhaltenen Lösungen wurden mit numerischen Ergebnissen verglichen.

РЕШЕНИЕ ЗАДАЧИ О ЗАТВЕРДЕВАНИИ СФЕРЫ МЕТОДОМ ДЕФОРМИРОВАННЫХ КООРДИНАТ

Аннотация—Для получения решения задачи о затвердевании насыщенной жидкости сферической формы используется метод деформированных координат. Решение справедливо для всей области и может применяться для случая движения фронта затвердевания к центру, где обычное возмущенное решение дает искажения. Температура стенки полагается постоянной. Однако этот метод можно использовать также при граничных условиях других типов. Свойства затвердевающего материала считаются постоянными. Для расширения области применения возмущенного решения используется нелинейное преобразование последовательности частичных сумм. Полученные решения сравниваются с численными результатами.